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# Bivariate Chebyshev maps of $\mathbf{C}^2$ and their dynamics

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## Abstract

We study the properties of bivariate (two-dimensional) Chebyshev maps  $T_n(x, y)$  from  $\mathbf{C}^2$  to  $\mathbf{C}^2$  and study the properties and dynamics of the maps.

(A) The properties of  $T_n$ .

(1) Solutions of  $T_n(x, y) = (0, 0)$  are obtained.

(2) A critical set  $\det(DT_n) = 0$  is written in a simple formula.

These properties are similar to those of Chebyshev maps of  $\mathbf{C}$ .

(B) The dynamics of  $T_n$ .

(1)  $T_n$  is strictly critically finite.

(2) Any periodic point of  $T_n$  is repelling.

(3) The exact form of the invariant probability measure  $\mu$  of maximal entropy associated with  $T_n$  is obtained.

(4) External rays for  $J_2(T_n)$  and foliations of  $J_1(T_n)$  are studied.

These properties are also similar to those of Chebyshev maps of  $\mathbf{C}$ .

## 1 Bivariate Chebyshev maps

The Chebyshev map is a typical chaotic map. Generalized Chebyshev maps are studied by several researchers, Koornwinder [1974], Lidle [1975], Veselov [1987] and Hoffman & Withers [1988].

In this paper, we study bivariate Chebyshev maps  $T_n$  from  $\mathbf{C}^2$  to  $\mathbf{C}^2$ ,  $n \in \mathbf{Z}$ .

$$T_n(x, y) = (g^{(n)}(x, y), g^{(n)}(y, x)).$$

This definition is due to [V]. Here  $g^{(n)}(x, y)$  is a generalized Chebyshev polynomial defined by Lidle [L].

Let

$$x = t_1 + t_2 + t_3, \quad y = t_1 t_2 + t_1 t_3 + t_2 t_3, \quad 1 = t_1 t_2 t_3.$$

Then

$$g^{(n)}(x, y) := t_1^n + t_2^n + t_3^n.$$

So

$$g^{(n)}(y, x) = (1/t_1)^n + (1/t_2)^n + (1/t_3)^n = g^{(-n)}(x, y).$$

For instance,

$$T_2(x, y) = (x^2 - 2y, y^2 - 2x),$$

$$T_3(x, y) = (x^3 - 3xy + 3, y^3 - 3xy + 3),$$

$$T_4(x, y) = (x^4 - 4x^2y + 2y^2 + 4x, y^4 - 4xy^2 + 2x^2 + 4y).$$

$\{g^n(x, y)\}$  satisfy the following recurrence equation:

$$g^{(n)}(x, y) = xg^{(n-1)}(x, y) - yg^{(n-2)}(x, y) + g^{(n-3)}(x, y).$$

First, we show a branch covering over  $\mathbb{C}^2$ .

The following diagram is commutative.

$$\begin{array}{ccc} (\mathbb{C} - \{0\})^2 & \xrightarrow{g_n} & (\mathbb{C} - \{0\})^2 \\ \downarrow \Psi & & \downarrow \Psi \\ \mathbb{C}^2 & \xrightarrow{T_n} & \mathbb{C}^2 \end{array}$$

$$\text{where} \quad g_n(u, v) = (u^n, v^n),$$

and

$$(x, y) = \Psi(u, v) = \left(u + v + \frac{1}{uv}, \frac{1}{u} + \frac{1}{v} + uv\right).$$

The covering map

$$\Psi : \mathbb{C}^2 - \Psi^{-1}(D) \rightarrow \mathbb{C}^2 - D$$

is a 6-sheeted covering map. Branch locus  $D$  of  $\Psi$  is written as

$$x^2y^2 - 4x^3 - 4y^3 + 18xy - 27 = 0.$$

In the case  $n = 2$ , Ueda[[Ue](#)] showed this diagram.

$T_n(x, y)$  restricted on  $\{x = \bar{y}\}$  is a Chebyshev polynomial defined by Koornwinder [[K](#)]

$$P_{n,0}^{-\frac{1}{2}}(z, \bar{z}) = e^{in\sigma} + e^{-in\tau} + e^{i(n\tau - n\sigma)}.$$

Set

$$z(\sigma, \tau) := e^{i\sigma} + e^{-i\tau} + e^{i(\tau - \sigma)} = u + iv.$$

The mapping

$$z : (\sigma, \tau) \rightarrow (u, v)$$

is a diffeomorphism from  $R$  onto  $S$ . See Koornwinder [K].

**Proposition 1.** *There are  $n^2$  solutions of  $T_n(x, y) = (0, 0)$ . All solutions lie in the closed domain  $S$  in  $\{x = \bar{y}\}$ . They are written in the  $(\sigma, \tau)$  coordinate.*

$$(1) \quad (\sigma, \tau) = \left( \frac{2(1+j+h)\pi}{3n}, \frac{2(1+2j+h)\pi}{3n} \right)$$

$$j = 0, 1, \dots, n-1, \text{ and } h = 0, 1, \dots, j.$$

$$(2) \quad (\sigma, \tau) = \left( \frac{2(2+j+h)\pi}{3n}, \frac{2(2+2j-h)\pi}{3n} \right)$$

$$j = 0, 1, \dots, n-2, \text{ and } h = 0, 1, \dots, j.$$

**Proof.** By definition,

$$T_n(x, y) = (g^{(n)}(x, y), g^{(n)}(y, x)).$$

$g^{(n)}(x, y)$  and  $g^{(n)}(y, x)$  are polynomials of degree  $n$  with no common components.

We can find  $n^2$  zeros on  $S$ . See Uchimura [Uc1].  $\square$

We see that the zeros of  $T_n$  and  $T_{n+1}$  "mutually separate each other".

Next we consider critical set of  $T_n(x, y)$ .

$$C_n := \{(x, y) \in \mathbb{C}^2 : \det(DT_n) = 0\}.$$

**Proposition 2.** *Let  $n \in \mathbb{Z}$ . Assume that*

$$x = t_1 + t_2 + t_3, \quad y = t_1 t_2 + t_1 t_3 + t_2 t_3, \quad t_1 t_2 t_3 = 1.$$

*Then*

$$\det(DT_n) = n^2 \frac{t_1^n - t_2^n}{t_1 - t_2} \cdot \frac{t_1^n - t_3^n}{t_1 - t_3} \cdot \frac{t_2^n - t_3^n}{t_2 - t_3}.$$

**Proof.**

$$\det(DT_n) = \det(D(T_n \circ \Psi)) / \det(D\Psi). \quad \square$$

The similar result is holds for generalized Chebyshev maps from  $\mathbb{C}^n$  to  $\mathbb{C}^n$ .

**Corollary 1.** *Any irreducible component of  $C_n$  is a rational curve of degree 2 or 4.*

**Proof.** From Proposition 2, we have

$$x = t + \epsilon^k t + \frac{1}{\epsilon^k t^2}, \quad \epsilon = e^{\frac{2\pi i}{n}}$$

$$y = \frac{1}{t} + \frac{1}{\epsilon^k t} + \epsilon^k t^2.$$

When  $\epsilon^k = -1$ , the degree of the rational curve is 2. □

We see that  $C_n$  and  $C_{n+1}$  "mutually separate each other", and

$$C_n \cap S \neq \emptyset \quad (S = J_2(T_n)).$$

Note that  $\{T_m : m \in \mathbb{Z}\}$  is a semigroup satisfying

$$T_m \circ T_n = T_{mn}.$$

## 2 Dynamics of Bivariate Maps

We study the dynamics of  $T_n(x, y)$ . Let

$$K(T_n) := \{(x, y) : \{T_n^m(x, y)\} \text{ is bounded for any } m\}.$$

In our setting we have the following proposition.

**Proposition 3.**

$$K(T_n) = \{|t_1| = |t_2| = 1\} = S \subset \{x = \bar{y}\}.$$

**Proof**

$$\begin{array}{ccc} (t_1, t_2) & \xrightarrow{g_n} & (t_1^n, t_2^n) \\ \downarrow \Psi & & \downarrow \Psi \\ (x, y) & \xrightarrow{T_n} & (g^{(n)}, g^{(-n)}) \end{array}$$

□

$f$  is called *critically finite* if each irreducible component of the critical set of  $f$  is periodic or preperiodic. Dihn and Sibony [DS] show that generalized Chebyshev maps are critically finite. Here using proposition 2, we give a direct proof.

**Proposition 4.**  $T_n$  is strictly critically finite.

**Proof.**

$$\begin{array}{ccccc} C_n & \xrightarrow{T_n} & T_n(C_n) & \xrightarrow{T_n} & T_n(C_n) \\ (t, \epsilon t) & & (t^n, t^n) & & (t^{n^2}, t^{n^2}) \end{array} \quad \square$$

Next we study the second Julia set  $J_2$  of  $T_n(x, y)$ .

**Proposition 5.** All periodic points of  $T_n$  lie on  $S$  and are equidistributed in  $S$ .

**Proof.** From [FS], we know that number of periodic points with period  $k$  equals  $n^{2k}$ . For the distribution of periodic points, see [Uc2].  $\square$

**Proposition 6.** *Any periodic point of  $T_n$  is repelling.*

To prove this proposition we consider the following function.

$$S_n := T_n | \{x = \bar{y}\} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$$

$$\text{e.g. } S_2(z) = z^2 - 2\bar{z} : (u, v) \mapsto (u^2 - 2u - v^2, 2uv + 2v).$$

**Lemma 1.** *Let  $p$  be a periodic point of  $S_n$ . Let  $\alpha$  and  $\beta$  be eigen values of  $DS_n(p)$ . Then*

$$|\alpha|, |\beta| > 1.$$

**Proposition 7.** *Let*

$$f(x, y) \in \mathbf{R}[x, y].$$

$$T(x, y) := (f(x, y), f(y, x)) : \mathbf{C}^2 \rightarrow \mathbf{C}^2.$$

$$t(z) := T | \{x = \bar{y}\} : \mathbf{R}^2 \rightarrow \mathbf{R}^2.$$

*Then*

$$U^{-1}DT(z, \bar{z})U = Dt(z),$$

*where*

$$U = \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}.$$

From Lemma 1 and Proposition 7, Proposition 6 follows.  $\square$

Next we study the invariant measure  $\mu$  of maximal entropy for  $T_n$ .

**Proposition 8.** *Under the above notation,*

$$\text{supp } \mu = S.$$

$$\mu = \left(\frac{2}{\pi}\right)^2 \frac{dx_1 dx_2}{\sqrt{-x^2 \bar{x}^2 + 4x^3 + 4\bar{x}^3 - 18x\bar{x} + 27}}.$$

$$(x = x_1 + ix_2)$$

This is an extension of invariant measure

$$\mu = \frac{1}{\pi} \frac{dx}{\sqrt{(x+1)(3-x)}}$$

for Chebyshev maps in one variable on  $[-1, 3]$ .

**Proof.** We prove this proposition in the following three steps.

(1) Briend and Duval [BD] shows that

$$\text{let } \mu_n := \frac{1}{d^{nk}} \sum_{f^n(y)=y, y \text{ repelling}} \delta_y,$$

then

$$\mu_n \rightarrow \mu \quad (\text{weak convergence}).$$

(2) From Proposition 5, we see that the periodic points are repelling and equidistributed in the triangle on the  $(s, t)$  plane (see [Uc2]).

(3) Pullback of Lebesgue measure under  $\phi$ . □

Next we consider the properties of external rays of  $T_n(x, y)$ . We use the definitions of external rays by Bedford and Jonsson [BJ]. We extend the map

$$T_n(x, y) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$$

$$\text{to } \hat{T}_n(x : y : z) : \mathbb{P}^2 \rightarrow \mathbb{P}^2.$$

Let  $\Pi := \mathbb{P}^2 - \mathbb{C}^2$  be the line at infinity.

Then

$$\hat{T}_n|_{\Pi} : (x : y : 0) \rightarrow (x^n : y^n : 0).$$

Therefore

$$J_{\Pi} = \{(x : y : 0) : |x| = |y|\} \simeq S^1.$$

The stable set of  $J_{\Pi}$  for  $T_n$  is defined by

$$W^s(J_{\Pi}, T_n) := \{x \in \mathbb{P}^2 : d(T_n^j x, J_{\Pi}) \rightarrow 0, \quad j \rightarrow \infty\}.$$

Bedford and Jonsson [BJ] state that there exists a Böttcher coordinate  $\Psi$  such that

$$\Psi : W^s(J_{\Pi}, f_n) \rightarrow W^s(J_{\Pi}, T_n)$$

conjugating  $f_n$  to  $T_n$ , where

$$f_n(x, y) = (x^n, y^n).$$

They also show that  $W^s(J_\Pi, T_n)$  is foliated by stable disks  $W_a$ . They define a local stable manifold  $W_{loc}^s(a)$ , ( $a \in J_\Pi$ ) and then a stable disk  $W_a \supset W_{loc}^s(a)$  and an external ray  $R(a, \theta)$ . They show that  $J_0(T_n) = J_1(T_n)$  is laminated by stable disks  $W_a$ .

Nakane [N] shows the following results on  $T_2(x, y)$ :

(1) The map  $\Psi$  defined by Ueda is essentially the inverse of Böttcher coordinate  $\phi$ .

$$\Psi(u, v) = \Psi(t, at), |t| > 1.$$

(2) The stable disk  $W_a$  is the set of points  $R(r, \phi, \theta)$

$$x = re^{-2\pi i\theta} + \frac{1}{r}e^{2\pi i(\theta-\phi)} + e^{2\pi i\phi},$$

$$y = re^{2\pi i(\phi-\theta)} + \frac{1}{r}e^{2\pi i\theta} + e^{-2\pi i\phi}, \quad a = e^{2\pi i\phi}, \quad (r > 1).$$

An external ray is written as

$$R(\phi, \theta) := \{R(r, \phi, \theta) : r > 1\}.$$

From this,

$$J_2 = S \subset \{x = \bar{y}\}.$$

(3) Each point  $z \in S$  is the landing point of exactly 1, 3, or 6 external rays if  $z$  is a cusp point on  $\partial S$ ,  $z$  is non-cusp point on  $\partial S$  or  $z \in \text{int}(S)$  respectively.

We can show that Nakane's results are also true for any  $T_n(x, y)$ ,  $n \neq 0$ .

Next we study the structure of foliations  $W_a$  of

$$J_1(T_n) = W^s(J_\Pi, T_n).$$

**Proposition 9.** *For any point  $z \in \text{int}(S)$ , there exist three stable disks  $W_a$  such that boundaries of these three disks intersect at  $z$ . At the point, two external rays on each  $W_a$  land from opposite directions.*

Metaphorically speaking, three mouths (stable disks) eat a sandwich (the second Julia set  $S$ ).

Two external rays  $R(\phi, \theta)$  and  $R(\phi, \phi - \theta)$  lie on the stable disk

$$W_a \quad (a = e^{2\pi i\phi}).$$



Two points  $R(r, \phi, \theta)$  and  $R(r, \phi, \phi - \theta)$  are "symmetrical" about  $\{x = \bar{y}\}$  in the following sense.

- (1) The midpoint of the segment  $\overline{R(r, \phi, \theta)R(r, \phi, \phi - \theta)}$  lies on the plane  $\{x = \bar{y}\}$ ,
- (2) The segment connecting two points is perpendicular to  $\{x = \bar{y}\}$ .

We compare the external rays of  $T_n(x, y)$  with those of Chebyshev map  $T_n(z)$  in one variable. The external rays  $T_n(z)$  is written as

$$R(r, \phi) : u = re^{2\pi i\phi} + \frac{1}{r}e^{2\pi i(-\phi)}, \quad (r > 1).$$

Clearly,

$$R(r, -\phi) : v = re^{2\pi i(-\phi)} + \frac{1}{r}e^{2\pi i\phi},$$

$$v = \bar{u}.$$

It is well-known that  $R(r, \phi)$  and  $R(r, -\phi)$  are "symmetrical" about the real axis. Note that symmetric group  $S_2$  acts on external rays of  $T_n(z)$ . On the other hand,  $S_3$  acts on external rays of  $T_n(x, y)$ .

Using the notations in Sect. 1, we can write

$$W^s(J_\Pi, T_n) = \{\Psi(t_1, t_2) : |t_1| = \frac{1}{|t_2|} > 1\}.$$

Then

$$C_n \cap W^s(J_\Pi, T_n) = \emptyset.$$

Lastly we consider periodic rays  $R(\phi, \theta)$  of  $T_n(x, y)$ .

**Proposition 10.** *If one periodic ray lands at the point  $z_0 \in S$ , all rays which land at  $z_0$  are all periodic with the same period.*

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